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SOME APPRECIATIVE REMARKS ON THE THEORY OF NUMBERS.

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In volume 10, 1903, of this journal the writer of the present note gave a brief list of appreciative remarks on the theory of groups with a view to furnishing an easy means to gain a knowledge of some important elements of this theory. The present brief list of appreciative remarks on the theory of numbers is intended to serve a similar purpose. It should be remembered that such extracts frequently require some modifications, but they direct attention to central truths of great value.

In the introduction to Reid's *Elements of the Theory of Algebraic Numbers*, 1910, Professor Hilbert asserts that "up to the present there is indeed no other science so highly praised by its devotees as is the theory of numbers." Some American students of mathematics might at first be inclined to attach little significance to these words in view of the fact that the number of devotees of this subject may be supposed to be very small. It should be observed that the number of those who study the theory of numbers, especially in Germany, is not insignificant. If it is remembered that Kummer's classes in this subject at the time of his greatest popularity in the University of Berlin, numbered at least 250,* and that classes of more than 100 are not uncommon now, in this leading German university, it is evident that conclusions as regards the number of devotees of this subject should not be based on conditions in our own universities.

In 1907 Minkowski began the preface of his book, entitled *Diophantische Approximation*, with the following words: "Integral numbers are the source (Urquell) of all mathematics. By this I understand not only the old view according to which the concept of continuity can also be deduced out of the consideration of discrete aggregates. I think much more about later results in using these words. The facts that the theory of the division of the circle dominates the theory of expotential functions and that the el-

* *Festschrift zur Feier dês 100 Geburtstages*, Edward Kummers, 1910, p. 15.

liptic functions may be comprehended by means of modular equations, inspire the confident belief that deepest relations in analysis are of an arithmetical nature." It may be of interest to note in this connection that the 15th reprint, 1910, of the Prospekt of the great German mathematical encyclopedia announces that part 5 of the Nachtraege to Band II will be devoted to relations between function theory and number theory, "Beziehungen zwischen Funktionentheorie und Zahlentheorie."

More than sixty years ago the noted French mathematician, Poinsot, expressed himself, in the *Journal de Mathêmatiques*, volume 10, page 2, as follows: It seems that the authors have regarded, since a long time, the theory of numbers as a queer speculation which is connected with nothing either in analysis or in geometry, and consequently offers to the intellect only truths which are more curious than useful. One finds scarcely any trace of it in the ordinary treatises on arithmetic and algebra. However, it is easy to see, after little reflection, that this transcendent arithmetic is the origin and source of real algebra. This is a truth which can be established by reason as I shall show directly, but which may also be proved, in a manner, by experience. For we believe that the little which is added from time to time to algebra comes from the little which is discovered at intervals in the science of the properties of numbers. An especial example of this kind is furnished by the felicitous relations connecting the algebraic solution of binomial equations of all degrees and the nature of prime numbers, according to which the circle can be divided into equal parts by means of the rule and the circle. This was an unexpected and very remarkable step which the theory of numbers furnished at the same time to algebra and to geometry.

"I believe that we shall at some time succeed to arithmetize all mathematical disciplines with the exception of geometry and mechanics; that is, to base them singly and solely on the concept of numbers in the most restricted sense, and hence cut off the modifications and extentions (namely the irrational and continuous quantities) of this concept, which were mainly due to applications to geometry and mechanics."* God made the integers; all the rest is the work of men, "Die ganzen Zahlen hat der lieve Gott gemacht, alles andere ist menschenwerk."† "It is a remarkable and suggestive fact that scarcely two hundred years after the discovery of the calculus, the higher mathematics has already exhibited a strong tendency to converge towards the oldest of all mathematical sciences, that of harmonious discontinuity — the theory of numbers."‡

While it is natural to suppose that specialists in number theory have made the most favorable remarks in regard to this subject it is not true that a high opinion of this subject is confined to such specialists. Among extreme specialists in synthetic geometry, Jacob Steiner occupies a promi-

* Kronecker, *Crelle Journal*, vol. 101 (1887), p. 338.

† Kronecker, Weber, *Mathematische Annalen*, vol. 43 (1893), p 15.

‡ Cole, *American Journal of Mathematics*, vol. 9 (1887), p. 46.

nent place; yet, according to an account by Lampe , Jacob Steiner advised his students especially to study the theory of numbers on the ground that this subject was eminently suited to cultivate acuteness. Steiner is said to have reprimanded severely a young student who wanted to devote himself exclusively to the study of systhetic geometry as Steiner had done himself. It is said that Steiner remarked that not all those who would say to him, Lord, Lord, could enter the Kingdom of Heaven.

The scientific developments of the fundamental facts of the theory of numbers are largely due to Gauss and they were first published in the classic Disquisitiones Arithmeticae, 1801. In a biographical sketch of Gauss entitled *Gauss zum Gedaechtniss*, 1856, Waltershausen stated that Gauss called mathematics the queen of the sciences and arithmetic the queen of mathematics. Although this science descends often to render service to astronomy and other natural sciences, yet it deserves under all conditions the first rank.

"Goepel was first attracted by the higher theory of numbers, like many of those who are chosen for mathematical speculations."† "There is a theory which has been equally useful to me in all my researches, namely, that of the groups formed by the linear substitutions. In fact, these substitutions play a preponderant role in the study of linear equations and in that of arithmetic forms. It is to this circumstance that one ought to attribute the interrelations, often unexpected, which I shall note later between the theory of numbers and the theory of Fuchsian functions,—theories which, moreover, do not at first appear to have any point of contact."‡ "From the problems which we have just examined we see that the three fundamental branches of mathematics, namely, the theory of numbers, algebra, and the theory of functions, are most intimately related; and I am convinced that the theory of analytic functions of several variables will make decided progress if one shall arrive at the discovery and the study of functions which, in the domain of any given algebraic numbers, play a role analogous to that played by the expotential functions in the domain of the rational numbers, and by the elliptic modular functions in the domain of quadratic imaginary numbers."§

It would be easy to extend this list of quotations very much but the object of the present note is attained if a sufficient number of quotations have been given to exhibit the importance and the bearing of the theory of numbers. Just as the engineer is inclined to hasten into his profession without a sufficient training in mathematics, so the student of mathematics is often tempted to proceed to fields for which he has been imperfectly prepared. While many mathematical subjects are mutually helpful, the question as to which should be regarded as the more fundamental is sometimes of great importance.

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* Lampe, *Bibliotheca Mathematica*, 3rd Series, vol. 1 (1900), p. 134.
† C. G. J. Jacobi, *Crelle Journal*, vol. 35 (1847), p. 313.
‡ Poincaré, *Note sur les travaux scientifiques* (1884), p. 7.
§ Hilbert, *Paris Intenational Mathematical Congress* (1900), p. 90.